








The Convolution Theorem: A Systems Approach

Convolution is a powerful tool for determining the output of a system to *any* input. The Convolution Theorem is developed here in a completely mathematical way. If the input to a system is $x(t)$, and the impulse response of that system is $h(t)$, then we can determine the output of the system, $y(t)$, from the integral:

$$y(t) = \int_0^t h(t-\lambda)x(\lambda)d\lambda$$

Note: the limits of integration could be $-\infty$ to $+\infty$ because $x(t)=0, t<0$ (so $x(\tau)=0, \lambda<0$), and $h(t)=0, t<0$ (so $h(t-\lambda)=0$ for $\lambda>t$).

Start with a linear time-invariant (LTI) system. The system has an input, $x(t)$, and an output, $y(t)$.	
Now consider the same system with impulse input, $\delta(t)$. The output is, by definition, the impulse response, $h(t)$.	
If we delay the input, the output is similarly delayed. (Time Invariance)	
If we scale the input, the output is similarly scaled. (Linearity)	
If we integrate the input the output is also integrated. (Also a property of linearity)	
By the sifting property of the impulse response we know that: $\int_0^t \delta(t-\lambda)x(\lambda)d\lambda = x(t)$	
Therefore the input to the system is $x(t)$.	
If the input is $x(t)$, the output is $y(t)$ (from top image).	
Equating the right sides of the last two images leads directly to the convolution theorem. $y(t) = \int_0^t h(t-\lambda)x(\lambda)d\lambda$	